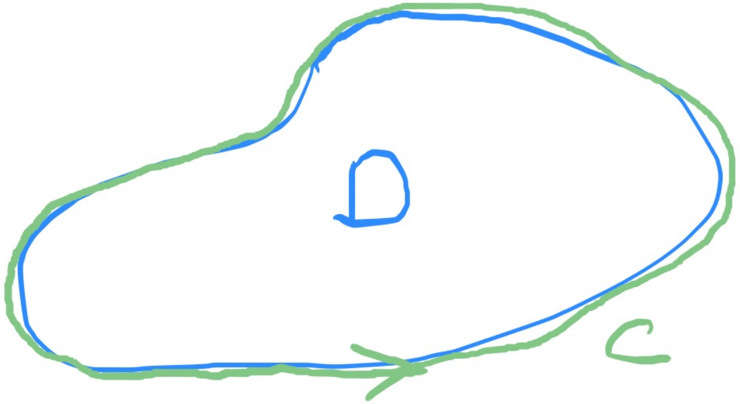
Ch. 8.1 Green's Theorem

relates double integral over a bounded domain $D \subset \mathbb{R}^2$ to a line integral over its boundary curve



C boundary curve
oriented counterclockwise

counter clockwise ~ positive orientation

<u>Green's Theorem</u>: $F: D \to \mathbb{R}^2$ vector field
$F(x,y) = (P(x,y), Q(x,y))$

$$\Rightarrow \boxed{\int_C F \cdot ds = \int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy}$$

<u>Example:</u> $F(x,y) = (x, xy)$

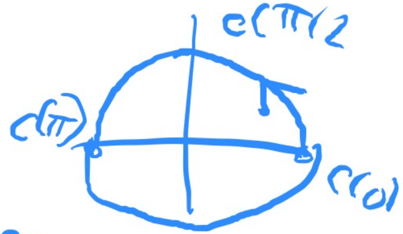
D unit disk

Verify Green's Theorem for this example!

boundary curve C = circle of radius 1

$c(t) = (\cos t, \sin t)$, $[0, 2\pi) \to \mathbb{R}^2$

check: counterclockwise orientation!



correct orientation!

$$\int_C F \cdot ds = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^{2\pi} F(\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (\underbrace{\cos t}_{x}, \underbrace{\cos t \sin t}_{x\ y}) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} -\cos t \sin t + \sin t \cos^2 t\, dt =$$

$$= \int_0^{2\pi} -\cos t \sin t + \cos^2 t \sin t \, dt$$

$$= -\frac{1}{2}\sin^2 t - \frac{1}{3}\cos^3 t \Big|_0^{2\pi} = \boxed{0}$$

by periodicity of sin and cos.

Recall: $P(x,y) = x$

$Q(x,y) = xy$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy = \iint_D y - 0 \, dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \, dy \, dx$$

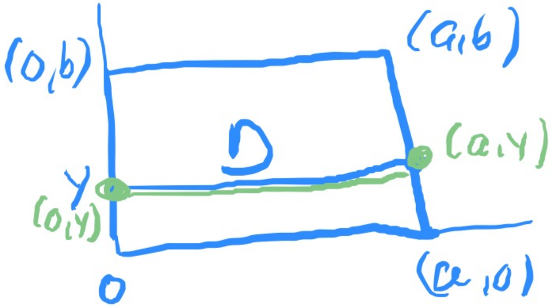
$$= \int_{-1}^{1} \frac{y^2}{2} \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1} \underbrace{1-x^2 - \left(-\sqrt{1-x^2}\right)^2}_{=0} dx =$$

$$= \boxed{0}$$

$=$

Proof of Green's Theorem for rectangle

(shows that Green's Theorem is a higher dimensional analog of Fundamental Theorem of Calculus)



$$F(x,y) = (P(x,y), Q(x,y))$$

$$\iint_D \frac{\partial Q}{\partial x}\, dx\, dy = \int_0^b \underbrace{\left[\int_0^a \frac{\partial Q}{\partial x}(x,y)\, dx \right]} dy$$

$$= \int_0^b Q(x,y) \Big|_{x=0}^{x=a} dy =$$

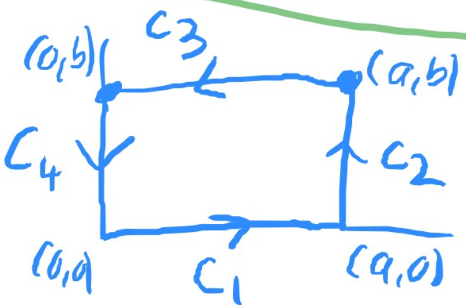
$$\boxed{\iint \frac{\partial Q}{\partial x}\, dx\, dy = \int_0^b Q(a,y) - Q(0,y)\, dy}$$

By same argument

$$\iint \frac{\partial P}{\partial y}\,dy\,dx = \int_0^a \left[\int_0^b \frac{\partial P}{\partial y}(\underset{\substack{\uparrow\\ \text{const}}}{x},y)\,dy\right]dx$$

$$= \int_0^a P(x,b) - P(x,0)\,dx$$

$$\Rightarrow \boxed{\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \int_0^b Q(a,y) - Q(0,y)\,dy - \int P(x,b) - P(x,0)\,dx}$$



		$c_i'(t)$
$c_1 : [0,a] \to \mathbb{R}^2$	$c_1(t) = (t,0)$	$(1,0)$
$c_2 : [0,b] \to \mathbb{R}^2$	$c_2(t) = (a,t)$	$(0,1)$
$c_3 : [0,a] \to \mathbb{R}^2$	$c_3(t) = (a-t,b)$	$(-1,0)$
$c_4 : [0,b] \to \mathbb{R}^2$	$c_4(t) = (0,b-t)$	$(0,-1)$

$$\int_C (P(x,y), Q(x,y)) \cdot ds =$$

$$= \int_{C_1} \left(P(c_1(t)), Q(c_1(t))\right) \cdot c_1'(t)\, dt$$

$$+ \dots$$

$$= \int_0^a \left(P(t,0), Q(t,0)\right) \cdot (1,0)\, dt$$

$$= \boxed{\int_0^a P(t,0)\, dt}$$

check: for c_2: $\displaystyle\int_{C_2} F \cdot ds = \int_0^b (P(a,t), Q(a,t)) \cdot (0,1)\, dt$

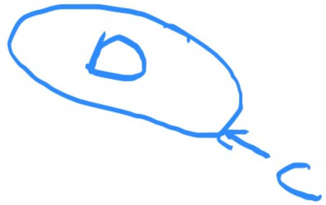
$$= \boxed{\int_0^b Q(a,t)\, dt}$$

for c_3: $\boxed{-\displaystyle\int_0^a P(t,a)\, dt}$

for c_4: $\boxed{\displaystyle\int_0^b - Q(0,t)\, dt}$

check:
sum of four integrals on this page
= integral on previous page
$\Rightarrow$ have proved Green's Theorem for rectangle

Application



Theorem: Let D and C be as before

$$\Rightarrow \boxed{\text{area}(D) = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_C (-y, x)\cdot ds}$$

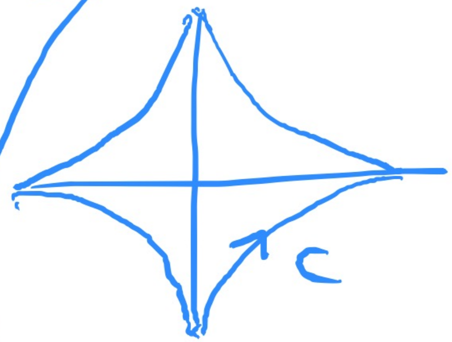
Example: Calculate area of region enclosed by hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$

parametrize C!

$$\Rightarrow \boxed{\begin{aligned} x^{1/3} &= a^{1/3}\cos t \\ y^{1/3} &= a^{1/3}\sin t \end{aligned}}$$

$$''\quad (x^{1/3})^2 + (y^{1/3})^2 = (a^{1/3})^2$$

$$\Rightarrow \quad \Rightarrow \quad \boxed{x = a\cos^3 t \qquad y = a\sin^3 t}$$



Parametrization c:

$$c(t) = (a\cos^3 t\ ,\ a\sin^3 t)$$

$$0 \leq t \leq 2\pi$$

$$\Rightarrow \quad \text{area} = \frac{1}{2}\int (-y, x)\cdot ds$$

$$= \frac{1}{2}\int (-a\sin^3 t,\ a\cos^3 t)\cdot \underbrace{(a\, 3\cos^2 t\,(-\sin t)}_{a\, 3\sin^2 t\cos t})\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} 3a^2 (\ \sin^4 t\cos^2 t + \cos^4 t\sin^2 t)\, dt$$

$$= \frac{3}{2}a^2 \int_0^{2\pi} \sin^2 t\cos^2 t\,(\underbrace{\sin^2 t + \cos^2 t}_{=1})\, dt$$

trig id. $\boxed{\sin t\cos t = \frac{1}{2}\sin 2t}$

$$= \frac{3}{2} a^2 \int_0^{2\pi} \frac{1}{4} \sin^2 2t \, dt$$

andere trig identity: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$= \frac{3}{8} a^2 \int_0^{2\pi} \frac{1}{2}(1 - \cos 4t) \, dt$$

$$= \frac{3}{8} a^2 \left(\frac{1}{2} t - \frac{1}{8} \sin 4t \Big|_0^{2\pi} \right)$$

$$= \frac{3}{8} a^2 \left(\frac{1}{2}(2\pi - 0) - \frac{1}{8}(0 - 0) \right)$$

$$= \boxed{\frac{3\pi}{8} a^2}$$